The Neoclassical Growth Model With Unequal Consumers and Concave Consumption Functions

Reto Foellmi, University of Zurich*

December 9, 2005

Abstract

Within the context of the neoclassical growth model I investigate the implications of (initial) endowment inequality when the rich have a higher marginal savings rate than the poor. It is shown that that the economy converges to a unique steady state. However, more unequal societies grow faster in the transition process, and therefore exhibit a higher speed of convergence. Further, there is divergence in consumption and lifetime wealth if the rich exhibit a higher intertemporal elasticity of substitution.

The uniqueness of a steady state contrasts with the implications of concave consumption functions in the Solow-Stiglitz model. When savings are exogenous, multiple steady states may emerge.

JEL classification: O40, D30, O10

Keywords: Marginal propensity to consume, income distribution, growth, concave consumption function.

*University of Zurich, Institute for Empirical Research in Economics, Bluemlisalpstrasse 10, CH-8006 Zürich, Tel: +41 44 634 37 26, Fax: +41 44 634 49 07, e-mail: rfoellmi@iew.unizh.ch.

1 Introduction

Rich people save more. Does this fact imply that inequality will increase for ever over time? And, how does this savings behavior affect the growth path? Tackling these questions is a priori a complex task: On the one hand, inequality affects capital accumulation when the marginal propensities to consume (MPC) differ. On the other hand, inequality itself changes through the accumulation process because savings rates differ and the factor prices change. It is the purpose of this paper to analyze this relationship within the context of the neoclassical growth model with perfect and complete markets.

Theoretical reasoning that savings propensities increases with wealth date back at least to Fisher (1930) and Keynes (1936). Carroll and Kimball (1996) show that when agents are subject to uninsurable risks or liquidity constraints the consumption function is concave expect for special cases. To make our point as simple as possible, we will analyze a model with full certainty but where the different MPC arise due to non-homothetic preferences. Looking at household data, it is a well established fact that rich people save more - in marginal and average terms - out of wealth or permanent income, see the recent paper by Dynan, Skinner, and Zeldes (2004) and their references. Partly due to poor data comparability the empirical picture is less conclusive on the aggregate level, the differences between studies are due to different data sets and different approaches to tackle the endogeneity problems. Although Schmidt-Hebbel and Servén (2000) and Li and Zou (2004) could not find a robust effect of inequality on saving, Cook (1995) and Smith (2001) found a positive effect of inequality on (private) saving rates.[1]

Stiglitz (1969) studied the dynamics of distribution when savings are exogenously given linear function of wealth. Chatterjee (1994), Caselli and Ventura (2001) and Bertola, Foellmi, and Zweimueller (2005, Chap. 3) study the same question in a

[1]The well-known studies of Barro (2000) and Forbes (2000) obtain a positive inequality growth relationship. This is also consistent with the view that inequality raises savings.

Ramsey model where the linear savings rule is a result of dynamic optimization. The impact of concave consumption functions for the evolution of inequality and growth was previously studied by Bourguignon (1981) and Schlicht (1975) in the context of the Solow-Stiglitz model. Bourguignon (1981) shows that multiple steady states may emerge which can be pareto-ranked. In this paper, convex savings are the result of a dynamic optimization with intertemporally separable preferences. Surprisingly, the analysis is much simplified: the steady state equilibrium is unique and independent of the initial distribution. This result implies that more unequal societies must exhibit a higher speed of convergence because they grow faster in the transition process.

The paper is structured as follows. Section 2 presents the model. Both the competitive equilibrium and the social planner's solution are analyzed. Section 3 then presents a numerical simulation and in the final section 4 the differences to the Bourguignon's model are discussed.

2 The model

2.1 Set-up

Preferences All consumers have the same intertemporal additive preferences and the same discount rate. The time horizon is infinite. Hence, the intertemporal utility function is given by

$$U_i = \int_0^\infty e^{-\rho t} u(c_i(t))dt \tag{1}$$

where $c_i(t)$ denotes consumption of individual i at date t. We assume that $u(\cdot)$ is twice continuously differentiable above some (subsistence) level $\bar{c} \geq 0$. We take the usual assumption that $u' > 0 > u''$, i.e. marginal utility is declining but the individual is non-satiated (at least over the relevant range). Further, the Inada conditions hold: $\lim_{c \to \bar{c}} u'(c) = \infty$. The first two assumptions imply that the elasticity of substitution is positive for all $c > \bar{c}$:

$$-\frac{u'(c)}{u''(c)c} > 0 \text{ for } c > \bar{c} \geq 0.$$

Individual factor endowments We assume that - at date 0 - household i is endowed with l_i units of labor, which is assumed to be constant over time, and $k_i(0)$ units of capital. We restrict the inequality in the way that each household is able to consume more than $\bar{c}$ in every period of time. We will come back to this assumption below. The number of households is constant. Hence total amount of labor L is also constant and we normalize it to one. Hence, the total amount of labor and capital in the economy is given by

$$\begin{aligned} K &\equiv \int_{\mathcal{N}} k_i(t) dP_i \\ 1 &\equiv \int_{\mathcal{N}} l_i dP_i \end{aligned}$$

where $\mathcal{N}$ denotes the set of families and dP_i the size of family i.

Technology and factor rewards The inputs labor and capital are used to produce a homogenous output good Y which can be both used for consumption and investment. Production takes place within a standard neoclassical production function with constant returns to scale. We assume that there is no technological progress, i.e. we focus on transitional dynamics only.[2]

$$Y(t) = F(K(t), 1) \equiv f(K(t))$$

The factors are rewarded their marginal products, hence the interest rate and the wage rate are given by

$$\begin{aligned} r(t) &= f'(K(t)) \qquad (2) \\ w(t) &= f(K(t)) - K(t) f'(K(t)) \end{aligned}$$

[2] As is well known, with positive growth we only get steady states if the intertemporal elasticity of substitution is constant, i.e. utility is CRRA.

and are uniquely determined by the current capital stock $K(t)$.

2.2 The decentralized equilibrium

Markets are perfect and complete. We assume that each household is able to consume more than $\bar{c}$. All individuals face the same factor prices, thus the household's income is given by $w(t)l_i + r(t)k_i(t)$. The evolution of individual wealth then reads $\dot{k}_i(t) = w(t)l_i + r(t)k_i(t) - c_i(t)$. Imposing the transversality condition we get the intertemporal budget constraint. The utility maximization problem of the consumer reads

$$\max_{\{c_i(t)\}} \int_0^\infty e^{-\rho t} u(c_i(t))dt \ \text{ s.t. } \int_0^\infty e^{-R(t)} c_i(t)dt \leq k_i(0) + \int_0^\infty e^{-R(t)} w(t) l_i dt$$

where $R(t) = \int_0^t r(s)ds$.

Differentiating the first order condition of this problem with respect to time we get the well known Euler equation

$$\dot{c}_i(t) = -\frac{u'(c_i(t))}{u''(c_i(t))} \left(r(t) - \rho \right). \tag{3}$$

Aggregating (3) we obtain the equation of motion for aggregate consumption

$$\dot{C}(t) = (r(t) - \rho) \int_{\mathcal{N}} -\frac{u'(c_i(t))}{u''(c_i(t))} dP_i. \tag{4}$$

Equation (4) already allows us to determine how inequality affects consumption growth and the savings rate.

Proposition 1 *If $-u'(c)/u''(c)$ is convex, i.e. the consumption function is concave, more unequal societies (measured in terms of wealth) have a higher savings rate and grow faster.*

Proof. A second order stochastic dominance shift in the c_i- distribution increases $\int_{\mathcal{N}} -\frac{u'(c_i(t))}{u''(c_i(t))} dP_i$ when $-u'(c)/u''(c)$ is convex. Individual consumption is monotone in wealth $k_i(0) + \int_0^\infty e^{-R(t)} w(t) l_i dt$. Hence, consumption growth in (4) is higher with a more unequal wealth distribution. ■

As a corollary note that $-u'(c)/u''(c)$ being concave would imply that more unequal societies save less. Finally, savings are independent of distribution when $-u'(c)/u''(c)$ is linear. This is the well known result that income distribution has no effect on accumulation when preferences take the HARA (hyperbolic risk aversion) form.

At the same time we are able to draw conclusions on the evolution of the consumption and the wealth distribution.

Proposition 2 *Consumption and wealth inequality increases (decreases) in a growing economy if the elasticity of substitution $-u'(c)/u''(c)c$ increases (decreases) in c.*

Proof. From (3) we see that the growth rate of individual consumption $\dot{c}_i/c_i$ increases in c_i when $-u'(c)/u''(c)c$ increases (decreases) in c. Wealth inequality moves pari passu with consumption inequality since consumption is monotone in wealth. ■

The condition on the evolution of inequality is not directly related to the concavity of the consumption function. The concavity of the consumption function is a statement on *marginal* propensities to save. Instead, the evolution of the wealth and consumption inequality is governed by differences in saving rates, i.e. the *average* propensities to save. To take an example, when the consumption function is linear but exhibits a positive axis intercept due to subsistence consumption inequality will widen over time. This is the case with Stone-Geary utility $u(c) = \ln(c - \bar{c})$, for example. Intuitively, the subsistence consumption level forces a poor individual to save only little today which implies that the subsequent growth rate of wealth and consumption is lower for the poor.

2.3 The social planner's problem

To determine the welfare properties and to prove the uniqueness of the transitional path it is useful to consider the social planner's problem. The planner assigns welfare weights ω_i to the individuals which are pinned down by the (initial) distribution of k_i and l_i in the decentralized optimum analyzed in the next section. Set up the current value Hamiltonian with $\{c_i(t)\}$ as control and $K(t)$ as state variable

$$H = \int_{\mathcal{N}} \omega_i u(c_i(t)) dP_i + \lambda_t \dot{K}_t$$

subject to the capital accumulation constraint (the output good can be used both for consumption and investment)

$$\dot{K}(t) = f(K(t)) - \int_{\mathcal{N}} c_i(t) dP_i \tag{5}$$

The first order conditions read

$$\omega_i u'(c_i(t)) - \lambda(t) = 0 \tag{6}$$

$$\rho\lambda(t) - \dot{\lambda}(t) = \lambda(t) f'(K(t)). \tag{7}$$

Proposition 3 *The equilibrium is unique and pareto-efficient.*

Proof. The first order conditions (6) and (7) and the capital accumulation equation (5) give the standard pair of differential equations, we omit time indices,

$$\begin{aligned} \frac{\dot{\lambda}}{\lambda} &= \rho - f'(K) \\ \dot{K} &= f(K) - \int_{\mathcal{N}} c(\omega_i, \lambda) dP_i \end{aligned} \tag{8}$$

where $c(\omega_i, \lambda)$ is implicitly defined by $\omega_i u'(c_i) = \lambda$. Figure 1 depicts equations (8) with K on the horizontal and λ on the vertical axis. The $\dot{\lambda} = 0$ locus is vertical at $f'(K) = \rho$, and the $\dot{K} = 0$ locus is monotonically decreasing as $c(\omega_i, \lambda)$ is decreasing in λ. The system has a unique saddle path with negative slope. Hence the policy function $\lambda(K)$ is uniquely determined. ■

Figure 1

Ad Existence:

The system possesses a unique steady state (c_i^*, K^*)

2.4 Steady State

As there is no technical progress, the economy will be in steady state when C, Y, and K are constant. Setting $\dot{\lambda} = 0$ and $\dot{K} = 0$ in (8) yields us the steady state value of the interest rate and the consumption level

$$\begin{aligned} r^* &= f'(K^*) = \rho \\ C^* &= f(K^*). \end{aligned}$$

Hence, we see that the steady state capital stock is unique and independent of the distribution. Intuitively, as long as the interest rate exceeds the discount rate, all individuals optimally choose an increasing consumption path although the rate of growth differs across agents. However, for any (separable) utility function (1) it is optimal to choose a constant consumption flow only if $r = \rho$.

2.5 Speed of convergence

We saw that all economies converge to the same steady state but unequal economies grow faster in the transitional process. To bring these two results together we must follow that more unequal societies exhibit a higher speed of convergence towards the steady state. To calculate the speed of convergence $\dot{K}(t)/\left(K(t) - K^*\right)$ we linearize the economy around its steady state

$$\frac{\dot{C}(t)}{C(t) - C^*} \cong \frac{\dot{K}(t)}{K(t) - K^*} \cong \mu \equiv \frac{1}{2}\left[\rho - \sqrt{\rho^2 - 4f''(K^*)\int_{\mathcal{N}} \frac{u'(c_i^*)}{u''(c_i^*)} dP_i}\right]. \qquad (9)$$

The derivation of equation (9) is shown in the appendix. The following proposition proves our intuition.

Proposition 4 *More unequal societies exhibit a higher speed of convergence.*

Proof. An increase in income dispersion increases $\int_{\mathcal{N}} -\frac{u'(c_i(t))}{u''(c_i(t))} dP_i$ and increases the absolute value of μ. ■

Along the same lines, we get the expressions for the evolution of aggregate consumption and capital stock around the steady state

$$\begin{aligned}\frac{C(t)-C^*}{C^*} &\cong \int_{\mathcal{N}} -\frac{u'(c_i^*)}{u''(c_i^*)c_i^*}\frac{c_i^*}{C^*}dP_i\frac{f''(K^*)}{\mu}e^{\mu t}\frac{K(0)-K^*}{K^*} \\ \frac{K(t)-K^*}{K^*} &\cong e^{\mu t}\frac{K(0)-K^*}{K^*}.\end{aligned}$$

This analysis was restricted to a neighborhood of the steady state. In particular, the consumption inequality is evaluated at its steady state level. Hence, the linearization does not allow for "feedback" effects of income distribution on growth and vice versa. To study the dynamics outside of steady state we therefore have to refer to numerical simulations which is done in section 3.

3 Calibration

To study the quantitative effects involved we perform a simple quantitative exercise. Let marginal utility be given by $u'(c) = (c^\gamma - \bar{c}^\gamma)^{-\sigma}$ where $\bar{c} > 0$ (subsistence consumption) and $\gamma > 1$. It is easy to show that the resulting consumption function is concave in wealth when the interest rate exceeds the rate of time preference. Furthermore, the elasticity of substitution $-u'(c)/u''(c)c$ is increasing in consumption as long as $\bar{c} > 0$. The preference parameters are chosen as $\rho = 0.02$, $\sigma = 2$, $\bar{c} = 1$, and $\gamma = 0.01$. The new parameters $\bar{c}$ and γ determine the concavity of the consumption function. The MPC will react more strongly to changes in wealth the higher $\bar{c}$ and γ are. The aggregate production function takes a Cobb-Douglas form, $Y = K^\alpha$. The capital share is given by $\alpha = 0.33$. Hence the steady states values of capital and consumption are given by $K^* = (\alpha/\rho)^{1/(1-\alpha)} = 65.6$ and $C^* = (K^*)^\alpha = 3.94$.

To simplify things further we assume that there are only two groups in the population: β poor and $1-\beta$ rich agents. According to Wolff (1998), the top 20% in the US population own about 80% of financial wealth. To match the (financial) wealth distribution in reality, let $\beta = 0.8$ be the group size of the poor and we choose the following individual wealth levels at date 0: $k_P(0) = 10$ and $k_R(0) = 110$. Hence, with this specification, the richest 20% own 73% of aggregate wealth. The aggregate capital stock equals $K(0) = 30$ or around 45% of its steady state value. The only free parameter left is the distribution of wage incomes (labor endowments). In the low inequality simulation we chose $l_P = 0.8$ (a poor individual earns 80% of average wage income) whereas in the high inequality case we set $l_P = 0.5$.

Table 1, *Figure* 2

How well can this simple model with perfect markets account for differences in savings rates? The difficulties in estimating cross country relationships between inequality and savings rates notwithstanding, Smith (2001) estimated that an increase in the Gini coefficient by one standard deviation or 10 percentage points results is associated with a 1.5% increase in the country's savings rate. In Table 1 we see that, with the values of the parameters chosen, an increase in the consumption Gini by 10 percentage points the savings rate increases by 0.6 - 1 percentage points,[3] with higher marginal effects for higher levels of inequality. Although no elements of uncertainty are present, the model is able to generate reasonable quantitative effects. Further, the simulation shows the evolution of inequality and in particular the influence of higher savings rates of the rich. The positive subsistence consumption $\bar{c} = 1$ forces the poor to choose a flat consumption path (see Figure 2) which results in a slow accumulation of assets. For the high inequality specification the poor's assets in steady state are even lower than at the starting date (see Table 1, last column).

[3] Note that we evaluate the savings rates at the starting point of the transition process. Obviously, as the economy moves closer to the steady state the savings rates decline and equal zero in steady state.

4 Discussion

The equilibrium outcome in the Ramsey model with convex savings stands in sharp contrast with Bourguignon's finding. With optimal savings and *infinite* horizons the equilibrium sequences of interest rates and wages are unique and pareto-efficient. This holds although inequality affects the transition path with a general utility function $u(c)$. Furthermore, there is no multiplicity in steady states any more. Bourguignon's (1981) analysis of the Solow model with convex savings suggests that the poor might indirectly gain from redistribution. More inequality raises savings and investment and therefore wages as the economy produces more capital intensive. This mechanism is the reason why the consumption levels of the poor and the rich are higher in the inegalitarian steady state than in the egalitarian one. Hence, the inegalitarian steady state is "pareto-dominant". (Of course such a comparison is not possible since there are no utility functions in the Solow model and the transitional process would have to be taken into account). In the Ramsey model, however, the equilibrium allocation is always pareto optimal. The reason lies in the fact markets are perfect and complete. The extreme differences in the outcomes are due to the finite horizon of agents as the Solow-Stiglitz model can be rationalized by an OLG economy with (warm glow) bequests. However, this conjecture needs to be explored.

References

Barro, Robert J. (2000), Inequality and Growth in a Panel of Countries, *Journal of Economic Growth* 5, 5-32.

Bertola, Giuseppe, Reto Foellmi, and Josef Zweimueller (2005), *Income Distribution in Macroeconomic Models*, Princeton University Press, forthcoming.

Bliss, Christopher (2004), Koopmans Recursive Preferences and Income Convergence, *Journal of Economic Theory* 117, 124-139.

Bourguignon, François (1981), Pareto Superiority of Unegalitarian Equilibria in Stiglitz's Model of Wealth Distribution with Convex Saving Function, *Econometrica* 49, 1469-1475.

Carroll, Christopher D. and Miles S. Kimball (1996), On the Concavity of the Consumption Function, *Econometrica* 64, 981-992.

Caselli, Francesco, and Jaume Ventura (2000), A Representative Consumer Theory of Distribution, *American Economic Review* 90, 909-926.

Chatterjee, Satyajit (1994), Transitional Dynamics and the Distribution of Wealth in a Neoclassical Growth Model, *Journal of Public Economics* 54, 97-119.

Cook, Christopher J. (1995), Savings Rates and Income Distribution: Further Evidence from LDCs, *Applied Economics* 27, 71-82.

Dynan, Karen E., Jonathan Skinner, and Stephen P. Zeldes (2004), Do the Rich Save More?, *Journal of Political Economy* 112, 397-444.

Fisher, Irving (1930), *The Theory of Interest*, New York: Macmillan.

Forbes, Kristin (2000), A Reassessment of the Relationship between Inequality and Growth, *American Economic Review* 90, 869-887.

Li, Hongyi and Heng-fu Zou (2004), Savings and Income Distribution, *Annals of Economics and Finance* 5, 245-270.

Lucas, Robert E. and Nancy L. Stokey (1984), Optimal Growth with Many Consumers, *Journal of Economic Theory* 32, 131-171.

Keynes, John Maynard (1936), *The General Theory of Employment, Interest, and Money,* Harcourt Brace Jovanovich Publishers, First Harbinger Edition 1964.

Schlicht, Ekkehart (1975), A Neoclassical Theory of Wealth Distribution, *Jahrbücher für Nationalökonomie und Statistik* 189, 78-96.

Schmidt-Hebbel, Klaus and Luis Servén (2000), Does income inequality raise aggregate saving?, *Journal of Development Economics* 61, 417-446.

Smith, Douglas (2001), International evidence on how income inequality and credit market imperfections affect private saving rates, *Journal of Development Economics* 64, 103-127.

Sorger, Gerhard (2002), On the Long-Run Distribution of Capital in the Ramsey Model, *Journal of Economic Theory* 105, 226-243.

Stiglitz, Joseph E. (1969), Distribution of Income and Wealth Among Individuals, *Econometrica* 37, 382-397.

Wolff, Edward N. (1998), Recent Trends in the Size Distribution of Household Wealth, *Journal of Economic Perspectives* 12(3), 131-150.

5 Appendix

To derive the speed of convergence, we take a first-order Taylor approximation around the steady state (where $f'(K) = \rho$). For the evolution of individual consumption (3) we get

$$\begin{aligned} \dot{c}_i &\cong \frac{\partial \dot{c}_i}{\partial c_i}\left[c_i - c_i^*\right] + \frac{\partial \dot{c}_i}{\partial K}\left[K - K^*\right] \\ &= f''(K)\frac{u'(c_i)}{u''(c_i)}\left[K - K^*\right]. \end{aligned}$$

By aggregation we get the evolution of aggregate consumption (note that $\dot{C} = \overset{\cdot}{C - C^*}$)

$$\overset{\cdot}{C - C^*} \cong f''(K)\int_{\mathcal{N}} \frac{u'(c_i)}{u''(c_i)} dP_i \left[K - K^*\right]. \tag{A1}$$

In the same way we approximate the capital accumulation equation $\dot{K} = f(K) - C$,

$$\overset{\cdot}{K - K^*} \cong \rho\left[K - K^*\right] - \left[C - C^*\right]. \tag{A2}$$

As (A1) and (A2) are linear in C and K, the growth rates of $[C - C^*]$ and $[K - K^*]$ coincide. The solution of this log linearized system is (9).

Table 1: Calibration

	High inequality	Low inequality	Representative Agent
Initial values			
Labor endowment l_P	0.5	0.8	1
Asset endowment $k_P(0)$	10	10	30
Asset endowment $k_R(0)$	110	110	30
Consumption of the poor $c_P(0)$	1.367	1.912	2.652
Consumption of the rich $c_R(0)$	7.320	5.430	2.652
Aggregate consumption $C(0)$	2.557	2.615	2.652
Consumption GINI	37.2	21.5	0
Savings rate	16.8%	14.9%	13.7%
Steady State			
Consumption of the poor c_P*	1.512	2.421	3.940
Consumption of the rich c_R*	13.799	10.018	3.940
Aggregate consumption $C*$	3.940	3.940	3.940
Assets of the poor k_P*	9.70	16.16	64.74
Assets of the rich k_R*	289.38	263.44	64.74

Figure 1: Phase Diagram

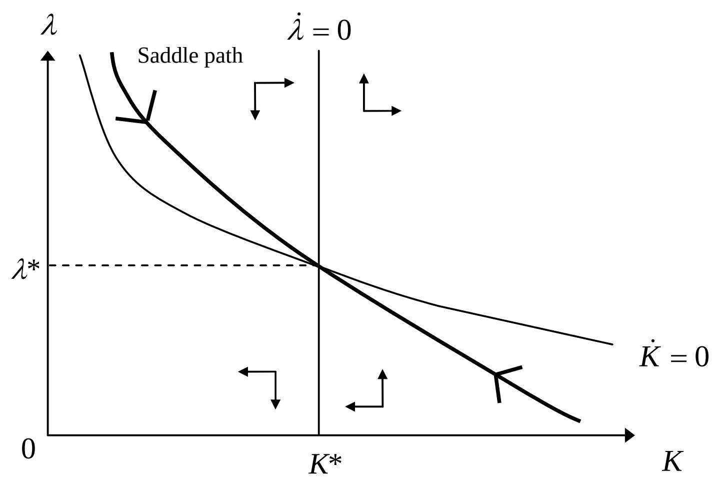

Figure 2: Dynamics of individual variables for low initial inequality